Coding Capacity and Error Exponent of the Poisson Channel with Time-Varying Parameters

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Abstract - The coding capacity of the Poisson channel with nonrandom periodic noise intensity is obtained for an average constraint and a periodic peak constraint on the encoder intensity. The case of almost periodic channel parameters is also treated. For these Poisson channels, the coding capacity equals the information capacity. Also, the coding capacity is the same with and without causal feedback. A random coding lower bound on the optimum error exponent is given for these classes of Poisson channels. These results are derived using work of Wyner for the Poisson channel with nontime-varying channel parameters.

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The coding capacity of the Poisson channel with nonrandom periodic noise intensity is obtained for an average constraint and a periodic peak constraint on the encoder intensity. The case of almost periodic channel parameters is also treated. For these Poisson channels, the coding capacity equals the information capacity. Also, the coding capacity is the same with and without causal feedback. A random coding lower bound on the optimum error exponent is given for these classes of Poisson channels. These results are derived using work of Wyner for the Poisson channel with nontime-varying channel parameters

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I. Introduction

The Poisson channel model addressed in this paper is represented in Figure 1. The channel out-

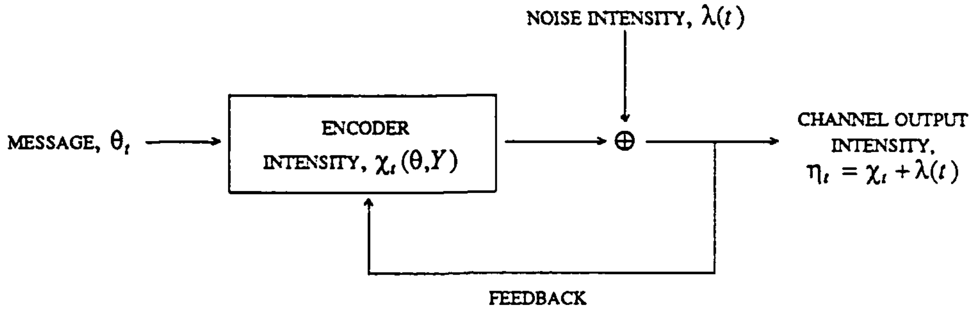

Figure 1. Poisson channel model.

put $Y=\{Y_t\}_{0\le t\le T}$ is a Poisson-type (simple) point process [10] directed by the stochastic intensity $\eta_t = \chi_t + \lambda(t)$. Y and the message process $\theta = \{\theta_t\}_{0\le t\le T}$ are defined on a common probability space $(\Omega, \mathbb{F}, P)$ with respective completed natural histories $\boldsymbol{F}_t^Y$ and $\boldsymbol{F}_t^{\theta}$. The message θ is encoded into the channel encoder output via the encoder intensity χ_t. $\chi_t = \chi_t(\theta, Y)$ is required to be an $\boldsymbol{F}_t^{\theta}\vee\mathbb{F}_t^{Y}$ -adapted encoding of the message process θ and channel output Y permitting causal message encoding and noiseless, nonanticipative, instantaneous feedback. The channel noise process N_t is a nonhomogeneous Poisson process with (nonnegative) intensity function $\lambda(t)$. $\lambda(t)$ is sometimes said to represent dark current in the channel model. We take the processes θ and N to be independent and impose, on the encoder intensity, a time-varying peak constraint

$$0 \le \chi_t \le c(t) \tag{1}$$

for all $t \in [0,T]$ where $c(t)$ is positive, bounded, and Lebesgue-measurable and an average constraint

$$E\left[\int_0^T \chi_t\, dt\right] \le k_0 T \tag{2}$$

for some positive constant k_0. The Poisson channel model is elsewhere referred to as the Poisson-type point process channel or as the direct detection photon channel. Further discussion of this channel model can be found in [2], [5], [6], [9], and [14] and the references cited therein.

The Poisson channel information capacity for time-varying channel parameters $\lambda(t)$, $c(t)$ is known, as is the channel coding capacity in the case of nontime-varying noise intensity and nontime-

varying peak constraint. The information capacity for the case of time-varying parameters was obtained in [6] by applying simple function approximation to Kabanov's capacity result [9] for the Poisson channel. Wyner [14] obtained the coding capacity for the nontime-varying parameter case by reducing the Poisson channel to a binary discrete memoryless channel. In this paper, we combine the method of approximation by simple functions and Wyner's result to give the coding capacity in the case where the channel parameters are time-varying. In particular, we address the case in which the noise intensity $\lambda(t)$ and the peak constraint function $c(t)$ are each periodic with periods T_λ and T_c respectively. Using a similar approach we also obtain the coding capacity for the case of almost periodic channel parameters. Periodic and almost periodic channel parameters are considered for their physical relevance and also to impose a degree of stationarity on the channel; the classes of channel parameters considered here ensure that the information capacity exists in the limit as $T \to \infty$.

II. CHANNEL CAPACITY DEFINITIONS

Coding capacity is defined within the context of codes, decoding schemes, and decoding error probability. A code (M,T,P_e) for the Poisson channel is a set of M equally likely waveforms $\chi_m(t)$, $t \in [0,T]$, $m=1,...,M$ satisfying the peak constraint

$$\chi_m(t) \le c(t), \qquad 0 \le t \le T$$

and the average constraint

$$\frac{1}{M}\sum_{m=1}^{M}\frac{1}{T}\int_0^T \chi_m(t)dt \le k_0 .$$

Let S_Y be the space of trajectories of Y on $[0,T]$. A decoding scheme is a mapping $D: S_Y \to \{1,2,...,M\}$. The error probability associated with D is

$$P_e = \frac{1}{M}\sum_1^M P\,\{D(Y_0^T) \ne m \mid \chi_m(\cdot)\}$$

where $Y_0^T \in S_Y$ denotes the path $\{Y_t\colon t \in [0,T]\}$. A code (M,T,P_e) has rate $R=(1/T)\ln M$. A code rate R is said to be achievable, for all $\varepsilon > 0$, there exists a code (M,T,P_e) whose parameters satisfy $M \ge e^{RT}$ with $P_e \le \varepsilon$ for T sufficiently large. The coding capacity $\mathcal{C}_{\text{CODING}}$ is the supremum of achievable rates.

To define channel information capacity let μ_θ, μ_Y, and $\mu_{\theta Y}$ be the marginal and joint measures induced by the message and output processes, θ and Y, on the path spaces S_θ, S_Y, and $S_\theta \times S_Y$. Write the induced product measure as $\mu_{\theta\times Y}$. Then, the average mutual information in θ and Y over the interval $[0,T]$ is

$$I^T[\theta,Y] = E\left[\ln\frac{d\mu_{\theta Y}}{d\mu_{\theta\times Y}}\right]$$

provided $\mu_{\theta Y} \ll \mu_{\theta\times Y}$; otherwise $I^T[\theta,Y]=\infty$. Expressions for the average mutual information specific to the Poisson channel are given in [8]. The channel information capacity is

$$\mathcal{C}_{\text{INFO}} = \sup_\theta \sup_\chi \frac{1}{T} I^T[\theta,Y]$$

where θ is any jointly measurable process on $[0,T]$ and $\chi_t = \chi_t(\theta,Y)$ is any $\boldsymbol{F}_t^\theta \vee \boldsymbol{F}_t^Y$-adapted mapping.

The information capacity of the Poisson channel with time-varying channel parameters is [6]

$$\mathbf{C}_{\text{INFO}} = \frac{1}{T}\int_0^T C_D(k_0,\lambda(t),c(t))dt \ ,$$

where $C_D(k_0,\lambda,c)$ is the information capacity found by Davis [5] for the Poisson channel with nontime-varying channel parameters $\lambda(t)=\lambda$ and $c(t)=c$. We have

$$C_D(x,y,z) = m(x,y,z)\phi(z,y) - \phi(zm(x,y,z),y) ,$$

$$\phi(x,y) = (x+y)\ln(x+y) - y\ln y ,$$

and $m(x,y,z)=r_m(x,z)\wedge r_o(y,z)$ is the minimum of $r_m(x,z)$ and $r_o(y,z)$ for $r_m(x,z)=x/z$ and

$$r_o(y,z) = \frac{1}{e}\frac{y}{z}\left[1+\frac{z}{y}\right]^{1+y/z} - \frac{y}{z} .$$

$r_o(\lambda(t),c(t))$ can be interpreted as the instantaneous optimum ratio of average to peak signal intensity. $r_m(k_0,c(t))$ is the instantaneous maximum ratio of average to peak signal intensity. In this connection, we note that

$$\frac{1}{e} \le r_o(\lambda(t),c(t)) \le \frac{1}{2}$$

for all $\lambda(t)$, $c(t)$, and k_0.

Fano's inequality provides an upper bound for the coding capacity in terms of the information capacity:

$$\mathbf{C}_{\text{CODING}} \le \limsup_{T\to\infty} \mathbf{C}_{\text{INFO}} .$$

In the Appendix we prove that, in the case where $\lambda(t)$ and $c(t)$ are periodic, the RHS is just the limit

$$\lim_{T\to\infty} \mathbf{C}_{\text{INFO}} = \frac{1}{T_\lambda T_c}\int_0^{T_\lambda}\int_0^{T_c} C_D(k_0,\lambda(t_\lambda),c(t_c))dt_c\,dt_\lambda . \qquad (3)$$

Thus, for the case of periodic channel parameters, the coding capacity is upper bounded by (3).

III. Coding Capacity for Periodic Channel Parameters

Our main result (Theorem 3) is that, for periodic channel parameters $\lambda(t)$ and (bounded) $c(t)$ with respective periods T_λ and T_c, the Poisson channel coding capacity is

$$\mathbf{C}_{\text{CODING}} = \frac{1}{T_\lambda T_c} \int_0^{T_\lambda} \int_0^{T_c} C_D(k_0, \lambda(t_\lambda), c(t_c)) dt_c \, dt_\lambda .$$

Thus, for periodic channel parameters, the coding capacity is the same as the channel information capacity [6] as $T \to \infty$.

We find the coding capacity with the aid of several lemmas. First, we construct simple periodic channel parameters $\bar{\lambda}(t)$, $\bar{c}(t)$. Let $A + x = \{t : t - x \in A\}$ and define

$$\bar{E}_j = \bigcup_{k=0}^{\infty} \{E_j + kT_\lambda\},$$

$$\bar{F}_i = \bigcup_{k=0}^{\infty} \{F_i + kT_c\}.$$

where $\{E_j\}$ and $\{F_i\}$ are each Lebesgue-measurable partitions of $[0, T_\lambda]$ and $[0, T_c]$ respectively. Then, we write

$$\bar{\lambda}(t) = \sum_{j=1}^{n_\lambda} \lambda_j 1_{\bar{E}_j}(t),$$

$$\bar{c}(t) = \sum_{i=1}^{n_c} c_i 1_{\bar{F}_i}(t).$$

Second, the channel with simple periodic parameters $\bar{\lambda}(t)$ and $\bar{c}(t)$ is transformed by changes of time into an ensemble of $n_\lambda n_c$ parallel Poisson subchannels (Lemma 1). Then, the coding capacity of the original Poisson channel is shown (Lemma 2) to be lower bounded by the sum of the $n_\lambda n_c$ coding capacities $\mathbf{C}^{ij}_{\text{CODING}}$ of the ensemble of parallel Poisson subchannels:

$$\mathbf{C}_{\text{CODING}} \geq \sum_{j=1}^{n_\lambda} \sum_{i=1}^{n_c} \mathbf{C}^{ij}_{\text{CODING}} .$$

The proof of Theorem 3 uses Eq. (3), Wyner's expression [14] for $\mathbf{C}^{ij}_{\text{CODING}}$, and the preceding lemmas to give the result for simple periodic channel parameters. To complete the proof, a limiting argument is used to extend the result to general (not necessarily simple) periodic $c(t)$ and $\lambda(t)$.

Lemma 1: Let

$$\tau_{ij} = \tau_{ij}(t) = \frac{T}{m_{ij}(T)} \int_0^t 1_{\bar{E}_j \cap \bar{F}_i}(s) ds \tag{4}$$

where

$$m_{ij}(T) = |\bar{E}_j \cap \bar{F}_i \cap [0,T]|.$$

The channel with time $\tau = \tau_{ij}$, message process $\bar{\theta}_\tau = \theta_t$, noise process $\bar{N}_\tau = N_t$, encoder output $\bar{X}_\tau = X_t$, and channel output $Y_\tau = X_\tau + N_\tau$ is a Poisson channel.

Proof: To be able to assert that a channel is Poisson, we need to check that 1) $Y_{\tau_{ij}}$, $N_{\tau_{ij}}$, and $X_{\tau_{ij}}$ are Poisson-type point processes, 2) causality is preserved, and 3) $\theta_{\tau_{ij}}$ and $N_{\tau_{ij}}$ are independent processes. But it is well-known [4], [8] that a change of time preserves the Poisson character of a process so requirement 1) is satisfied. $\tau = \tau_{ij}(t)$ is nondecreasing so 2) is satisfied. Also, requirement 3) is obviously satisfied so the assertion of the lemma is valid.

The transformed time $\tau = \tau_{ij}$ is indexed by $i = 1,...,n_c$, $j = 1,...,n_\lambda$. Thus the original channel is transformed into an ensemble of $n_\lambda n_c$ channels. Each of these $n_\lambda n_c$ channels reflects a portion of the events occurring in the original channel and is therefore thought of as a subchannel of the original channel. The ijth subchannel mirrors events occurring in the original Poisson channel during the time $\bar{E}_j \cap \bar{F}_i$. Scaling the time t causes the subchannel intensities to be scaled; the new intensities are $\lambda_{ij} = \lambda_j m_{ij}(T)/T$ and $\chi_t^{ij} = \chi_t m_{ij}(T)/T$. Scaling the encoder intensity in this way is equivalent to scaling the encoder intensity constraints. The constraints in the ijth subchannel are

$$0 \le \chi_t^{ij} \le \frac{c_i m_{ij}(T)}{T},$$

and

$$E[\int_0^T \chi_t^{ij} dt] \le \frac{k_0 m_{ij}(T)}{T} T.$$

Note that because of the choice of time transformation and the periodicity of $\lambda(t)$ and $c(t)$, the parameters of each subchannel are constant w.r.t. time. We shall want to decompose the Poisson channel into parallel subchannels for the case of $t \in [0,\infty)$. Therefore, the limiting scale factor $m_{ij}(T)/T$ for $T \to \infty$ is

needed. We show in Appendix 2 that

$$\lim_{T\to\infty} \frac{m_{ij}(T)}{T} = \frac{|E_j|}{T_\lambda}\frac{|F_i|}{T_c}.$$

Lemma 2: Let $\mathbb{C}^{ij}_{\text{CODING}}$ be the coding capacity of the Poisson subchannel formed by the change of time (4). Then

$$\mathbb{C}_{\text{CODING}} \geq \sum_{j=1}^{n_\lambda}\sum_{i=1}^{n_c} \mathbb{C}^{ij}_{\text{CODING}}.$$

Proof: Let R_{ij}, $1 \leq i \leq n_c$, $1 \leq j \leq n_\lambda$ be achievable rates for the ensemble of parallel Poisson subchannels. Then there exist codes (M_{ij}, T, P_e^{ij}) such that, for T sufficiently large, $M_{ij} \geq e^{R_{ij}T}$ and $P_e^{ij} \leq \varepsilon$. Combination of these codes gives a code (M, T, P_e) on the original channel with

$$M = \prod_{i,j} M_{ij}$$

and

$$P_e = 1 - \prod_{i,j}(1 - P_e^{ij}).$$

The rate $R = (1/T)\ln M = \sum_{i,j} R_{ij}$ of this code is achievable since

$$M = \prod_{i,j} M_{ij} \geq \prod_{i,j} e^{R_{ij}T} = e^{RT}$$

and, for all $\varepsilon_1 > 0$, we can make $P_e \leq \varepsilon_1$ by choosing T large enough that $P_e^{ij} \leq 1 - (1-\varepsilon_1)^{1/n}$. Since the supremum of all achievable rates is no less than the supremum restricted to achievable rates of combined codes, the result is proved.

Theorem 3: Consider the Poisson channel with nonrandom periodic noise intensity $\lambda(t)$ and peak- and average-constrained encoder intensity, as in (1) and (2), such that the peak function $c(t)$ is periodic and bounded. This channel has coding capacity

$$\mathbb{C}_{\text{CODING}} = \frac{1}{T_\lambda T_c}\int_0^{T_\lambda}\int_0^{T_c} C_D(k_0, \lambda(t_\lambda), c(t_c))\,dt_c\,dt_\lambda.$$

Proof: We have (Appendix 2) that

$$\lim_{T\to\infty} \frac{m_{ij}(T)}{T} = \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c}$$

so, from Lemma 1 and [14],

$$\mathbb{C}^{ij}_{\text{CODING}} = C_D(k_0 \lim_{T\to\infty} \frac{m_{ij}(T)}{T}, \lambda_j \lim_{T\to\infty} \frac{m_{ij}(T)}{T}, c_i \lim_{T\to\infty} \frac{m_{ij}(T)}{T})$$

$$= C_D(k_0 \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c}, \lambda_j \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c}, c_i \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c})$$

$$= \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c} C_D(k_0, \lambda_j, c_i).$$

For clarity write $\mathbb{C}_{\text{CODING}}(\lambda, c)$ for the coding capacity of the Poisson channel with parameters $\lambda(t)$, $c(t)$. Then, using Lemma 2,

$$\mathbb{C}_{\text{CODING}}(\bar{\lambda}, \bar{c}) \geq \sum_{j=1}^{n_\lambda} \sum_{i=1}^{n_c} \frac{|E_j|}{T_\lambda} \frac{|F_i|}{T_c} C_D(k_0, \lambda_j, c_i)$$

$$= \frac{1}{T_\lambda T_c} \int_0^{T_\lambda} \int_0^{T_c} C_D(k_0, \bar{\lambda}(t_\lambda), \bar{c}(t_c)) dt_c \, dt_\lambda .$$

Let $\bar{\lambda}_n(t)$ belong to a sequence of simple functions converging downward to $\lambda(t)$ and let $\bar{c}_n(t)$ belong to a similar sequence converging upward to $c(t)$. Since, for each n, $\lambda(t) \leq \bar{\lambda}_n(t)$ and $c(t) \geq \bar{c}_n(t)$, we have (Appendix 3)

$$\mathbb{C}_{\text{CODING}}(\lambda, c) \geq \mathbb{C}_{\text{CODING}}(\bar{\lambda}_n, \bar{c}_n), \quad n = 1, 2, 3, \ldots$$

Then,

$$\mathbb{C}_{\text{CODING}}(\lambda, c) \geq \frac{1}{T_\lambda T_c} \int_0^{T_\lambda} \int_0^{T_c} C_D(k_0, \bar{\lambda}_n(t_\lambda), \bar{c}_n(t_c)) dt_c \, dt_\lambda$$

and, by Fatou's Lemma [11],

$$\mathbb{C}_{\text{CODING}}(\lambda, c) \geq \frac{1}{T_\lambda T_c} \int_0^{T_\lambda} \int_0^{T_c} C_D(k_0, \lambda(t_\lambda), c(t_c)) dt_c \, dt_\lambda .$$

This and (3) proves the result.

Wyner [14] showed that the coding capacity of the Poisson channel with constant parameters is not increased by feedback. The foregoing proof preserves this feature of Wyner's result. Hence we find that the coding capacity for the case of periodic channel parameters is the same with or without feedback.

IV. Coding Capacity for Almost Periodic Channel Parameters

Definition: A real function $f(t)$ defined on the nonnegative real line is almost periodic if, for any $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ with the property that any interval in the nonnegative real line of length $l(\varepsilon) > 0$ contains at least one point ξ such that

$$|f(t+\xi) - f(t)| < \varepsilon, \qquad 0 \le t < \infty.$$

This is the definition of almost periodic functions given by Bohr [1] (or see [3].) It is easily checked that the function $f(t) = \sin(t) + \sin(\pi t)$ is almost periodic but not periodic. Thus we are motivated to consider almost periodic channel parameters of the Poisson channel. We note that not all periodic functions are almost periodic [3]. In particular, almost periodic functions are bounded and uniformly continuous. Thus periodic channel parameters cannot be treated as a special case of the Poisson channel with almost periodic parameters.

We mention some properties of almost periodic functions which will be needed. Proofs can be found in [3].

Property 1: Suppose the function $\Phi(z_1, ..., z_n)$ of n real variables is uniformly continuous over its domain and let the functions $f_1(t), ..., f_n(t)$ be almost periodic. Then the composition $\Phi(f_1(t), ..., f_n(t))$ is almost periodic.

Property 2: Let $f(t)$ be almost periodic. Then the limit $\lim\limits_{T \to \infty} \frac{1}{T} \int_0^T f(t)\,dt$ exists.

Property 3: Almost periodic functions are bounded.

Theorem 4: Suppose a Poisson channel has almost periodic noise intensity $\lambda(t)$ and an almost periodic peak function $c(t)$. Then the channel coding capacity is the limit

$$C_{\text{CODING}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T C_D(k_0, \lambda(t), c(t))\,dt\,. \tag{5}$$

Proof: The proof hinges on the properties of almost periodic functions quoted above. By Property 3, $c(t)$ is bounded so [6]

$$\mathbf{C}_{\mathrm{INFO}} = \frac{1}{T}\int_0^T C_D(k_0, \lambda(t), c(t))dt .$$

By Property 1, $C_D(k_0, \lambda(t), c(t))$ is almost periodic. Property 2 then assures us that $\mathbf{C}_{\mathrm{INFO}}$ exists in the limit as $T \to \infty$. Therefore

$$\mathbf{C}_{\mathrm{CODING}} \le \lim_{T \to \infty} \frac{1}{T}\int_0^T C_D(k_0, \lambda(t), c(t))dt . \tag{6}$$

By Property 3, $\lambda(t)$ and $c(t)$ are bounded. Therefore, define

$$M_\lambda = \max_{t \ge 0} \lambda(t), \qquad M_c = \max_{t \ge 0} c(t).$$

Define, also, the simple functions

$$\bar{\lambda}_n(t) = \frac{M_\lambda}{2^n}\left\lceil \frac{2^n}{M_\lambda}\lambda(t) \right\rceil ,$$

$$\bar{c}_n(t) = \frac{M_c}{2^n}\left\lfloor \frac{2^n}{M_c} c(t) \right\rfloor .$$

We can write

$$\bar{\lambda}_n(t) = \sum_{j=1}^{2^n} 1_{E_j}(t)\lambda_j ,$$

$$\bar{c}_n(t) = \sum_{i=1}^{2^n} 1_{F_i}(t)c_i$$

where

$$\lambda_j = \frac{jM_\lambda}{2^n}, \quad E_j = \{t : \bar{\lambda}_n(t) = \lambda_j\} ,$$

$$c_i = \frac{(i-1)M_c}{2^n}, \quad F_i = \{t : \bar{c}_n(t) = c_i\} .$$

Define $m_{ij}(T) = |[0,T] \cap E_j \cap F_i|$. The limits

$$\alpha_{ij} = \lim_{T \to \infty} \frac{m_{ij}(T)}{T}$$

exist. For i, j such that $\alpha_{ij} \neq 0$, define the changes of channel time

$$\tau_{ij}(t) = \frac{1}{\alpha_{ij}} \int_0^t 1_{E_j \cap F_i}(s) ds .$$

These changes of channel time transform the channel with parameters $\bar{\lambda}_n(t)$, $\bar{c}_n(t)$ into (up to) 2^{2n} Poisson subchannels - each with constant channel parameters. $\alpha_{ij} = 0$ corresponds to a subchannel with a peak constraint function which is identically zero; such subchannels have zero capacity and can be ignored. $C_D(\cdot,\cdot,0) = 0$ so we have

$$\mathcal{C}_{\text{CODING}}(\bar{\lambda}_n, \bar{c}_n) \geq \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} C_D(\alpha_{ij} k_0, \alpha_{ij} \lambda_j, \alpha_{ij} c_i)$$

$$= \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \alpha_{ij} C_D(k_0, \lambda_j, c_i)$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} C_D(k_0, \lambda_j, c_i) m_{ij}(T)$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T C_D(k_0, \bar{\lambda}_n(t), \bar{c}_n(t)) dt .$$

$\bar{\lambda}_n(t)$, $\bar{c}_n(t)$ were defined so that $\lambda(t) \leq \bar{\lambda}_n(t)$ and $c(t) \geq \bar{c}_n(t)$. Therefore (Appendix 3),

$$\mathcal{C}_{\text{CODING}}(\lambda, c) \geq \mathcal{C}_{\text{CODING}}(\bar{\lambda}_n, \bar{c}_n)$$

for all n. Then, using Moore's theorem [8] (Appendix 1),

$$\mathcal{C}_{\text{CODING}}(\lambda, c) \geq \lim_{n \to \infty} \mathcal{C}_{\text{CODING}}(\bar{\lambda}_n, \bar{c}_n)$$

$$\geq \lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T C_D(k_0, \bar{\lambda}_n(t), \bar{c}_n(t)) dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \lim_{n \to \infty} \int_0^T C_D(k_0, \bar{\lambda}_n(t), \bar{c}_n(t)) dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T C_D(k_0, \lambda(t), c(t)) dt .$$

This and (6) combine to complete the proof.

Observations made in the last section with respect to coding capacity for periodic channel parameters apply as well to the almost periodic case: the coding capacity is the same with and without feedback and the coding capacity is the same as the information capacity in the limit as $T \to \infty$,

$$\mathcal{C}_{\text{CODING}} = \lim_{T \to \infty} \mathcal{C}_{\text{INFO}}.$$

Also, it is clear that expressions for the coding capacity can be given in cases of mixed channel parameter types: periodic $c(t)$ and almost periodic $\lambda(t)$ and vice versa.

The reader has perhaps noted the similarity of the approaches used to obtain the coding capacity in the case of periodic channel parameters and in the case of almost periodic channel parameters. This similarity suggests that there is a class of parameters which contains both periodic and almost periodic parameters such that the same basic approach could be used to obtain the coding capacity. The Stepanoff-almost periodic functions [12], [13] (or see [3]) contain both periodic and almost periodic functions. For channel parameters of this or some other general class, we conjecture that

$$\mathcal{C}_{\text{CODING}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T C_D(k_0, \lambda(t), c(t))\, dt$$

and, also,

$$\mathcal{C}_{\text{CODING}} = \lim_{T_c \to \infty} \lim_{T_\lambda \to \infty} \frac{1}{T_c T_\lambda} \int_0^{T_c} \int_0^{T_\lambda} C_D(k_0, \lambda(t_\lambda), c(t_c))\, dt_c\, dt_\lambda .$$

V. RANDOM CODING BOUND

The optimal error exponent for the Poisson channel was derived by Wyner [14] for the case of nontime-varying channel parameters. In his derivation, he used a random coding exponent to lower bound the optimal error exponent. Upper bounds were obtained by "sphere-packing" and "fixed composition code" arguments. He then combined these upper bounds to generate a "straight line" upper bound on the optimal error exponent. This last upper bound coincides with the random coding lower bound, thus exactly determining the optimal error exponent. It appears that Wyner's approach would also serve to derive the optimal error exponent for cases of time-varying channel parameters; however, the derivation would be long. Therefore, in treating the case of time-varying parameters, we just use Wyner's random coding bound to give a lower bound on the optimal error exponent. This bound then leads to an upper bound on the overall error probability. In deriving the lower bound on the optimal error exponent for the case of time-varying parameters, we actually only use Wyner's random coding lower bound. Therefore we refer to this lower bound as a random coding bound.

Recall that, for a given decoding scheme D, the Poisson channel probability of decoding error is

$$P_e = \frac{1}{M}\sum_{m=1}^{M} P\,\{D\,(Y_0^T \neq m \,|\chi_m(\cdot)\}\,.$$

A probability of decoding error P_e is achievable if there exists a code (M,T,P_e) whose rate R is achievable. For fixed M and T, define $P_e^*(M,T)$ to be the infimum of all achievable error probabilities P_e. For achievable rates R, $0 \le R < C$, the optimal error exponent (also called the reliability function [7]) is defined to be

$$E(R) = \limsup_{T\to\infty} \frac{-\ln P_e^*(M,T)}{T}$$

where $M = \lceil e^{RT} \rceil$. For large T we can write

$$P_e^*(M,T) = e^{-E(R)T + o(T)}.$$

The optimal error exponent derived by Wyner for the Poisson channel with nontime-varying channel parameters λ, c is expressed in terms of the following functions:

$$\tau(x) = \left[1 + \frac{c}{\lambda}\right]^{1/(1+x)} - 1\,,$$

$$q^*(x) = \frac{k_0}{c} \wedge \left\{\frac{1}{\tau(x)}\left[\frac{c}{\lambda(1+x)\tau(x)}\right]^{1/x} - \frac{1}{\tau(x)}\right\},$$

$$R^*(x) = \lambda\left[1+q^*(x)\tau(x)\right]^x \frac{(1+\tau(x))q^*(x)}{1+x}\ln(1+c/\lambda)$$

$$-\lambda\left[1+q^*(x)\tau(x)\right]^{x+1}\ln(1+q^*(x)\tau(x)),$$

$$E_1(x,y) = y + \frac{\lambda}{c}\left[1-(1+q^*(x)\tau(x))^{x+1}\right].$$

Also, we define $q^*(0)$ to be the limit [14, Appendix 1]

$$q^*(0) = \lim_{\rho\to 0} q^*(\rho) = \frac{k_0}{c} \wedge \left[\frac{\lambda}{c}\left[1+\frac{c}{\lambda}\right]^{1+\lambda/c} - \frac{\lambda}{c}\right].$$

For nontime-varying, constant parameters, λ, c, the Poisson channel information and coding capacities are identical; we use $\mathbb{C}$ to denote their common value. Then

$$E(R) \equiv \begin{cases} R_0 - R\,, & 0 \le R \le R_c \\ cE_1(\rho,q^*(\rho)) - \rho R\,, & R_c < R < \mathbb{C} \end{cases}$$

where $\rho \in [0,1]$ is an implicit function of R through $R = R^*(\rho)$, $R_c = R^*(1)$ is the channel "critical rate", and

$$R_0 = cE_1(1,q^*(1)) = c\left[\frac{k_0}{c}\wedge\frac{1}{2}\right]\left[1-\left[\frac{k_0}{c}\wedge\frac{1}{2}\right]\right]\left[\sqrt{1+\lambda/c} - \sqrt{\lambda/c}\right]^2$$

is the "cutoff rate" of the channel. Where necessary to show explicitly the dependence of these various quantities on λ, c, and k_0, we shall use the notation

$$\tau = \tau(\lambda,c),$$

$$q^*(x) = q^*(x;\lambda,c,k_0),$$

$$R^*(x) = R^*(x;\lambda,c),$$

$$E_1(x,y) = E_1(x,y;\lambda,c),$$

$$R_c = R_c(\lambda,c,k_0),$$

$$R_0 = R_0(\lambda,c,k_0).$$

Note that, for $\alpha > 0$,

$$\tau(\alpha\lambda,\alpha c) = \tau(\lambda,c),$$

$$q^*(x;\alpha\lambda,\alpha c,\alpha k_0) = q^*(x;\lambda,c,k_0),$$

$$R^*(x;\alpha\lambda,\alpha c) = \alpha R^*(x;\lambda,c),$$

$$E_1(x,y;\alpha\lambda,\alpha c) = E_1(x,y;\lambda,c),$$

$$E_W(\alpha R;\alpha\lambda,\alpha c,\alpha k_0) = \alpha E_W(R;\lambda,c,k_0),$$

$$R_c(\alpha\lambda,\alpha c,\alpha k_0) = \alpha R_c(\lambda,c,k_0),$$

$$R_0(\alpha\lambda,\alpha c,\alpha k_0) = \alpha R_0(\lambda,c,k_0).$$

We seek a lower bound on $E(R)$ for the Poisson channel with periodic channel parameters. First consider periodic simple channel parameters

$$\bar{\lambda}(t) = \sum_{j=1}^{n_\lambda} \lambda_j 1_{\bar{E}_j}(t),$$

$$\bar{c}(t) = \sum_{i=1}^{n_c} c_i 1_{\bar{F}_i}(t).$$

as in Section III. By changes of channel time, the Poisson channel with parameters $\bar{\lambda}(t)$, $\bar{c}(t)$ can be transformed into $n_\lambda n_c$ parallel Poisson subchannels. This is not necessarily the only way to operate the channel. Therefore

$$P_{e,o}^*(M,T) \le P_{e,p}^*(M,T)$$

where $P_{e,o}^*$ is the infimum of achievable error probabilities for the Poisson channel with parameters $\bar{\lambda}(t)$, $\bar{c}(t)$ and $P_{e,p}^*$ is the infimum of achievable error probabilities for the parallel combination of $n_\lambda n_c$ Poisson subchannels. Equivalently,

$$E(R) \ge E^p(R)$$

where $E(R)$ is the optimal error exponent of the original channel and $E^p(R)$ is the optimal error exponent of the parallel combination of subchannels. If each of the Poisson subchannels in the parallel combination is operated as a discrete memoryless channel à Wyner then

$$P_{e,p}^*(M,T) \le P_{e,p,DMC}^*(M,T),$$

$$E^p(R) \ge E^{p,DMC}(R)$$

where $P_{e,p,DMC}(M,T)$ is the infimum of achievable error probabilities for the parallel combination of discrete memoryless subchannels and $E^{p,DMC}(R)$ is the corresponding optimal error exponent. Also

$$E^{p,DMC}(R) \geq E_r^{p,D.\ C}(R)$$

where $E_r^{p,DMC}(R)$ is the random coding exponent of the parallel combination of discrete memoryless channels. Using the decomposition for $E_r^{p,DMC}(R)$ given in [7] and passing to the limit as in [14, Section III], we obtain

$$E(R) \geq \begin{cases} \sum_{i,j} [R_0(\alpha_{ij}\lambda_j,\alpha_{ij}c_i,\alpha_{ij}k_0) - R_{ij}], & 0 \leq R_{ij} \leq R^*(1;\alpha_{ij}\lambda_j,\alpha_{ij}c_i),\ \text{all } i,j \\ \sum_{i,j} \alpha_{ij}c_i E_1(\rho,q_{ij}^*(\rho);\alpha_{ij}\lambda_j,\alpha_{ij}c_i) - \rho R_{ij}, & \alpha_{ij}R^*(1;\lambda_j,c_i) \leq R_{ij} \leq \alpha_{ij}R^*(0;\lambda_j,c_i)\ \text{ all } i,j \end{cases}$$

where $q_{ij}^*(\rho) = q^*(\rho;\alpha_{ij}\lambda_j,\alpha_{ij}c_i,k_0) = q^*(\rho;\lambda_j,c_i,\alpha_{ij}k_0)$, where R_{ij} is the rate in the ijth Poisson subchannel, and where

$$\alpha_{ij} = \frac{|E_j|}{T_\lambda}\frac{|F_i|}{T_c}.$$

For channel parameters $\lambda(t)$, $c(t)$, define

$$\bar{R}_c(\lambda,c) = \frac{1}{T_c T_\lambda}\int_0^{T_c}\int_0^{T_\lambda} R^*(1;\lambda(t_\lambda),c(t_c))dt_\lambda dt_c .$$

Let $\mathbb{C}(\bar{\lambda},\bar{c}) = \mathbb{C}_{\text{CODING}} = \mathbb{C}_{\text{INFO}}$ be the capacity of the Poisson channel with parameters $\bar{\lambda}(t)$, $\bar{c}(t)$. It is a simple calculation to show that

$$\mathbb{C} = \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} \alpha_{ij}R^*(0;\lambda_j,c_i).$$

Also,

$$R = \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} R_{ij}$$

so, for $\bar{R}_c(\bar{\lambda},\bar{c}) \leq R \leq \mathbb{C}$, ρ is an implicit function of R through

$$\begin{aligned} R &= \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} R^*(\rho,\alpha_{ij}\lambda_j,\alpha_{ij}c_i) \\ &= \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} \alpha_{ij}R^*(\rho,\lambda_j,c_i) \\ &= \frac{1}{T_c T_\lambda}\int_0^{T_c}\int_0^{T_\lambda} R^*(\rho,\bar{\lambda}(t_\lambda),\bar{c}(t_c))dt_\lambda dt_c . \end{aligned}$$

Therefore

$$E(R) \ge \begin{cases} \bar{R}_0(\bar{\lambda},\bar{c}) - R\,, & 0 \le R \le \bar{R}_c(\bar{\lambda},\bar{c}) \\ \bar{E}_1(\rho;\bar{\lambda},\bar{c}) - \rho R\,, & \bar{R}_c(\bar{\lambda},\bar{c}) \le R \le C \end{cases} \tag{7}$$

where

$$\bar{R}_0(\bar{\lambda},\bar{c}) = \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} \alpha_{ij} R_0(\lambda_j, c_i, k_0)$$

$$= \frac{1}{T_c T_\lambda} \int_0^{T_c}\int_0^{T_\lambda} R_0(\bar{\lambda}(t_\lambda),\bar{c}(t_c),k_0)\,dt_\lambda dt_c$$

and where

$$\bar{E}(\rho;\bar{\lambda},\bar{c}) = \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} \alpha_{ij} c_i E_1(\rho, q_{ij}(\rho); \alpha_{ij}\lambda_j, \alpha_{ij} c_i)$$

$$= \sum_{i=1}^{n_c}\sum_{j=1}^{n_\lambda} \alpha_{ij} c_i E_1(\rho, q_{ij}(\rho); \lambda_j, c_i)$$

$$= \frac{1}{T_c T_\lambda} \int_0^{T_c}\int_0^{T_\lambda} \bar{c}(t_c) E_1(\rho q^*(\rho,\bar{\lambda}(t_\lambda),\bar{c}(t_c),k_0);\bar{\lambda}(t_\lambda),\bar{c}(t_c))\,dt_\lambda dt_c\,.$$

Based on the form of (7), $\bar{R}_0(\bar{\lambda},\bar{c})$ and $\bar{R}_c(\bar{\lambda},\bar{c})$ are identified, respectively, as the cutoff rate and critical rate of the Poisson channel with parameters $\bar{\lambda}(t)$, $\bar{c}(t)$.

Consider nonnegative, periodic, Lebesgue-measurable channel parameters $\lambda(t)$, $c(t)$ with $c(t)$ bounded. Let $E(R)$ be the optimal error exponent of the Poisson channel with these parameters. Let $\bar{\lambda}_n(t)$ belong to a sequence of periodic simple functions converging upward to $\lambda(t)$ and let $\bar{c}_n(t)$ belong to a similar sequence converging downward to $c(t)$. $\bar{c}_n(t)$ is chosen to be a bounded function for each n. Then, for each n,

$$E(R) \ge E^n(R)$$

where $E^n(R)$ is the optimal error exponent for the Poisson channel with parameters $\bar{\lambda}(t)$, $\bar{c}(t)$. Therefore, taking the limit as $n \to \infty$,

$$E(R) \ge E_r(R;\lambda,c) \equiv \begin{cases} \bar{R}_0(\lambda,c) - R\,, & 0 \le R \le \bar{R}_c(\lambda,c) \\ \bar{E}_1(\rho;\lambda,c) - \rho R\,, & \bar{R}_c(\lambda,c) \le R \le C(\lambda,c) \end{cases}$$

where $\mathbb{C}(\lambda,c)$ is the capacity of the channel with parameters $\lambda(t)$, $c(t)$, $\bar{R}_0(\lambda,c)$ is the channel cutoff rate, and $\bar{R}_c(\lambda,c)$ is the channel critical rate. Because of the way it was derived using random coding exponent results, $E_r(R;\lambda,c)$ is called the random coding exponent for the channel. Although we have not made the calculation, we would be surprised if a similar random coding exponent could not be given for the channels with almost periodic parameters. Also, although it has not been proven, it is reasonable to expect that, in fact,

$$E(R) = E_r(R;\lambda,c)$$

for Poisson channels with periodic or almost periodic parameters as is true for the case of nontime-varying channel parameters.

APPENDIX 1

For convenience of notation, the parameter k_0 is supressed throughout this appendix. In particular, we shall write $C(y,z) = C_D(x,y,z)$.

Theorem: Suppose $\lambda(t)$ and $c(t)$ are nonegative and periodic with respective periods T_λ and T_c. Also suppose $c(t)$ is bounded. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T C(\lambda(t),c(t))dt = \frac{1}{T_\lambda T_c}\int_0^{T_c}\int_0^{T_\lambda} C(\lambda(t_\lambda),c(t_c))dt_\lambda dt_c .$$

Proof: Let $\{E_{nj},\ 1 \le j \le m < \infty\}$ and $\{F_{ni},\ 1 \le i \le l < \infty\}$ be Lebesgue-measurable partitions of $[0,T_\lambda]$ and $[0,T_c]$, respectively (e.g. the E_{nj} are Lebesgue-measurable subsets of $[0,T]$ and the union of the E_{nj} is $[0,T]$.) m and l depend on n. Let $A + x = \{t : t-x \in A\}$ and define

$$\bar{E}_{nj} = \bigcup_{k=0}^{\infty}\{E_{nj} + kT_\lambda\},$$

$$\bar{F}_{ni} = \bigcup_{k=0}^{\infty}\{F_{ni} + kT_c\}.$$

Let $\{\tilde{\lambda}_n\}$ and $\{\tilde{c}_n\}$ be sequences of simple functions converging pointwise to $\lambda(t)$ and $c(t)$ respectively with

$$\tilde{\lambda}_n(t) = \sum_{j=1}^{m}\lambda_{nj}1_{\bar{E}_{nj}}(t),$$

$$\tilde{c}_n(t) = \sum_{i=1}^{l}c_{ni}1_{\bar{F}_{ni}}(t),$$

where $0 \le \lambda_{nj} < \infty$, $0 \le c_{ni} < \infty$ for all i,j,n. By the Dominated Convergence Theorem [11], we have

$$\frac{1}{T}\int_0^T C(\lambda(t),c(t))dt = \lim_{n\to\infty}\frac{1}{T}\int_0^T C(\tilde{\lambda}_n(t),\tilde{c}_n(t))dt . \tag{A1.1}$$

This convergence is uniform in T - as we show below - so, by Moore's theorem [8],

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T C(\lambda(t),c(t))dt = \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T C(\tilde{\lambda}_n(t),\tilde{c}_n(t))dt$$

$$= \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\sum_{j=1}^{m}\sum_{i=1}^{l}C(\lambda_{nj},c_{ni})|\bar{E}_{nj}\cap\bar{F}_{ni}\cap[0,T]|$$

$$= \lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{l}C(\lambda_{nj},c_{ni})\lim_{T\to\infty}\frac{|\bar{E}_{nj}\cap\bar{F}_{ni}\cap[0,T]|}{T}$$

$$= \lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{l}C(\lambda_{nj},c_{ni})\frac{|E_{nj}|}{T_\lambda}\frac{|F_{ni}|}{T_c}$$

$$= \lim_{n\to\infty}\frac{1}{T_\lambda T_c}\int_0^{T_\lambda}\int_0^{T_c} C(\tilde{\lambda}_n(t_\lambda),\tilde{c}_n(t_c))\,dt_c\,dt_\lambda$$

$$= \frac{1}{T_\lambda T_c}\int_0^{T_\lambda}\int_0^{T_c} C(\lambda(t_\lambda),c(t_c))\,dt_c\,dt_\lambda .$$

In the string of six equalities above, the fourth equality is justified by the result in Appendix 2. To complete the proof, it only remains to show that the convergence in (A1.1) is uniform in T. We write

$$\left|\frac{1}{T}\int_0^T C(\tilde{\lambda}_n(t),\tilde{c}_n(t))dt - \frac{1}{T}\int_0^T C(\lambda(t),c(t))dt\right|$$

$$\le \frac{1}{T}\int_0^T |C(\tilde{\lambda}_n(t),\tilde{c}_n(t)) - C(\tilde{\lambda}_n(t),c(t))|dt$$

$$+ \frac{1}{T}\int_0^T |C(\tilde{\lambda}_n(t),c(t)) - C(\lambda(t),c(t))|dt . \qquad \text{(A1.2)}$$

Now,

$$|C(\lambda,c_1)-C(\lambda,c_2)| \le |C(0,c_1)-C(0,c_2)| \le |c_1-c_2| ,$$

so we have the bound

$$\frac{1}{T}\int_0^T |C(\tilde{\lambda}_n(t),\tilde{c}_n(t)) - C(\tilde{\lambda}_n(t),c(t))|dt \le \frac{1}{T}\int_0^T |\tilde{c}_n(t) - c(t)|dt .$$

Consistent with its earlier assigned properties, the sequence $\{\tilde{c}_n(t)\}$ may be chosen such that, for every $\varepsilon>0$, there exists an n_0 such that, for all $n>n_0$,

$$|\tilde{c}_n(t) - c(t)| < \varepsilon$$

on $[0,T]-A$, $|A|<\varepsilon$. This is Egoroff's theorem [11]. $c(t)$ is assumed to be bounded; let M be an upper bound. Then

$$\frac{1}{T}\int_0^T |C(\tilde{\lambda}_n(t),\tilde{c}_n(t)) - C(\tilde{\lambda}_n(t),c(t))|dt \le \frac{1}{T}[(T-\varepsilon)\varepsilon+\varepsilon M] \le \varepsilon + \varepsilon\frac{M}{T} . \qquad \text{(A1.3)}$$

Let $B=\{t\in[0,T]\colon \lambda(t)>L\}$ for $L>0$. The second integral on the RHS of (A1.2) can be bounded as follows:

$$\frac{1}{T}\int_0^T |C(\tilde{\lambda}_n(t),c(t))dt - C(\lambda(t),c(t))|\,dt$$

$$\le \frac{1}{T}\int_{[0,T]-B} |C(\tilde{\lambda}_n(t),c(t)) - C(\lambda(t),c(t))|\,dt$$

$$+ \frac{1}{T}\int_B |C(\tilde{\lambda}_n(t),c(t)) - C(\lambda(t),c(t))|\,dt$$

$$\le \frac{1}{T}\int_{[0,T]-B} D(0,c(t))\,|\tilde{\lambda}_n(t)-\lambda(t)|\,dt + \frac{1}{T}\int_B C(L,c(t))\,dt$$

where $D(x,y)$ is the absolute value of $\partial C(x,y)/\partial x$. The last inequality above follows from the fact that, for all y, $D(\cdot,y)$ is a bounded decreasing function over the interval $[0,\infty)$. $D(0,\cdot)$ is an increasing function over the interval $[0,\infty)$ so $D(0,c(t))\le D(0,M)$. Thus,

$$\frac{1}{T}\int_0^T |C(\bar{\lambda}_n(t),c(t))dt - C(\lambda(t),c(t))|\,dt$$

$$\leq \frac{D(0,M)}{T}\int_{[0,T]-B} |\bar{\lambda}_n(t)-\lambda(t)|\,dt + \frac{1}{T}\int_B C(L,c(t))\,dt\,. \qquad \text{(A1.4)}$$

Once again using Egoroff's Theorem,

$$\int_{[0,T]-B} |\bar{\lambda}_n(t)-\lambda(t)|\,dt \leq (T-|B|-\varepsilon)\varepsilon + \varepsilon L\,.$$

Using this in (A1.4) and then combining the result with (A1.3) in (A1.2) gives

$$\left|\frac{1}{T}\int_0^T C(\bar{\lambda}_n(t),\bar{c}_n(t))dt - \frac{1}{T}\int_0^T C(\lambda(t),c(t))dt\right|$$

$$\leq \varepsilon + \varepsilon\frac{M}{T} + \frac{D(0,M)}{T}[(T-|B|-\varepsilon)\varepsilon + \varepsilon L] + \frac{|B|}{T}C(L,M)\,.$$

The above bound holds for all $L>0$ and, in particular, for all L arbitrarily large. $|B|\to 0$ and $C(L,M)\to 0$ as $L\to\infty$. Choose $L = 1/\sqrt{\varepsilon}$ and consider only $T>T_o$ for some fixed $T_o>0$. Then

$$\left|\frac{1}{T}\int_0^T C(\bar{\lambda}_n(t),\bar{c}_n(t))dt - \frac{1}{T}\int_0^T C(\lambda(t),c(t))dt\right|$$

$$\leq \varepsilon + \varepsilon\frac{M}{T_o} + D(0,M)\varepsilon + \frac{D(0,M)}{T_o}\sqrt{\varepsilon} + o(\varepsilon)\,.$$

Thus, for all $T>T_o$, the convergence in (A1.1) is uniform in T.

APPENDIX 2

Theorem: Let $m(T) = |\bar{E} \cap \bar{F} \cap [0,T]|$ where

$$\bar{E} = \bigcup_{k=0}^{\infty} \{E + kT_\lambda\}, \quad \bar{F} = \bigcup_{k=0}^{\infty} \{F + kT_c\}$$

and where E and F are Lebesgue-measurable subsets of $[0,T_\lambda]$ and $[0,T_c]$ respectively. Then

$$\lim_{T\to\infty} \frac{m(T)}{T} = \frac{|E|}{T_\lambda} \frac{|F|}{T_c}.$$

Proof: Without loss of generality, take $T_\lambda < T_c$. Define $\bar{E}(\rho) = \bar{E} - \rho$ for each $\rho \in [0,T_\lambda]$. Let Q be the set of $\rho \in [0,T_\lambda]$ for which there exists a T_ρ such that, for all $t \geq 0$,

$$1_{\bar{E}(\rho)\cap\bar{F}}(t + T_\rho) = 1_{\bar{E}\cap\bar{F}}(t).$$

Q is dense in $[0,T_\lambda]$. Thus, for each $\rho \in [0,T_\lambda]$, there exists a T'_ρ such that

$$1_{\bar{E}(\rho)\cap\bar{F}}(t + T'_\rho) = 1_{\bar{E}\cap\bar{F}}(t)$$

for all $t \in X_T \subset [0,T]$ where $|[0,T] - X_T| < 1$. This follows from the fact (Littlewood's first principle [11]) that the set $\bar{E}(\rho) \cap \bar{F} \cap [0,T]$ is very nearly a finite union of open intervals. For each $\rho \in [0,T]$, we have

$$\lim_{T\to\infty} \frac{|\bar{E}(\rho) \cap \bar{F} \cap [0,T]|}{T} = \lim_{T\to\infty} \left[\frac{1}{T} \int_0^{T'_\rho} 1_{E(\rho)\cap F}(t)dt + \frac{T - T'_\rho}{T} \frac{1}{T - T'_\rho} \int_{T'_\rho}^{T} 1_{E(\rho)\cap F}(t)dt \right]$$

$$= \lim_{T\to\infty} \frac{1}{T - T'_\rho} \int_{T'_\rho}^{T} 1_{E(\rho)\cap F}(t)dt$$

$$= \lim_{T\to\infty} \frac{1}{T - T'_\rho} \int_0^{T-T'_\rho} 1_{E(\rho)\cap F}(s + T'_\rho)ds$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^{T} 1_{E(\rho)\cap F}(s + T'_\rho)ds$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^{T} 1_{E\cap F}(s)ds$$

$$+ \lim_{T\to\infty} \frac{1}{T} \int_{[0,T]-X_T} \left[1_{E(\rho)\cap F}(s + T'_\rho) - 1_{E\cap F}(s) \right] ds.$$

Now

$$\left| \int_{[0,T]-X_T} \left[1_{E(\rho)\cap F}(s + T'_\rho) - 1_{E\cap F}(s) \right] ds \right| \leq 2|[0,T] - X_T| \leq 2$$

so

$$\lim_{T\to\infty} \frac{|\bar{E}(\rho)\cap\bar{F}\cap[0,T]|}{T} = \lim_{T\to\infty} \frac{1}{T}\int_0^T 1_{\bar{E}\cap\bar{F}}(t)\,dt$$

$$= \lim_{T\to\infty} \frac{m(T)}{T}\,.$$

Let R be a random variable, uniformly distributed over the interval $[0,T_\lambda]$. By the Bounded Convergence theorem [9],

$$\lim_{T\to\infty} \frac{m(T)}{T} = E\left[\lim_{T\to\infty} \frac{|\bar{E}(R)\cap\bar{F}\cap[0,T]|}{T}\right]$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^{T_\lambda} 1_{\bar{E}(R)\cap\bar{F}}(t)\,dR\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^{T_\lambda} 1_{\bar{E}(R)}(t)\,dR\; 1_{\bar{F}}(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{|\bar{E}|}{T_\lambda} 1_{\bar{F}}(t)\,dt$$

$$= \frac{|\bar{E}|}{T_\lambda}\frac{|\bar{F}|}{T_c}\,.$$

APPENDIX 3

We use the notation $\mathcal{C}_{\text{CODING}}(\lambda,c)$ to denote the coding capacity of the Poisson channel with parameters $\lambda(t)$, $c(t)$. In this appendix we show that, if $c_1(t)$ dominates $c_2(t)$ and $\lambda_1(t)$ is dominated by $\lambda_2(t)$, then

$$\mathcal{C}_{\text{CODING}}(\lambda_1,c_1) \geq \mathcal{C}_{\text{CODING}}(\lambda_2,c_2). \tag{A3.1}$$

This fact is obvious from an examination of the formulae given for the coding capacity. However, since (A3.1) is used in the derivation of those formulae, an independent proof is required. We begin with two lemmas.

Lemma: Let $\lambda(t)$ be a nonnegative simple function. For the Poisson channel with parameters $\lambda(t)$, $c_o(t)$,

$$\inf_{c\in A}\mathcal{C}_{\text{CODING}}(\lambda,c) = \sup_{c\in B}\mathcal{C}_{\text{CODING}}(\lambda,c) = \mathcal{C}_{\text{CODING}}(\lambda,c_o)$$

where A is the set of all simple functions $c(t)$ dominating $c_o(t)$ and B is the set of all nonnegative simple functions $c(t)$ dominated by $c_o(t)$.

Proof: The first equality is proved by decomposing the Poisson channel into parallel subchannels. The proof is straightforward so we omit it. The proof of the second equality is a "proof by contradiction."

For $c \in A$, $\mathcal{C}_{\text{CODING}}(\lambda,c) \geq \mathcal{C}_{\text{CODING}}(\lambda,c_o)$ thus

$$\inf_{c\in A}\mathcal{C}_{\text{CODING}}(\lambda,c) \geq \mathcal{C}_{\text{CODING}}(\lambda,c_o).$$

Suppose

$$\inf_{c\in A}\mathcal{C}_{\text{CODING}}(\lambda,c) > \mathcal{C}_{\text{CODING}}(\lambda,c_o). \tag{A3.2}$$

λ is simple so there exists $c_1 \in A$ and $c_2 \in B$ such that

$$\mathcal{C}_{\text{CODING}}(\lambda,c_1) - \mathcal{C}_{\text{CODING}}(\lambda,c_2) < \Delta$$

for any $\Delta > 0$. Choose

$$\Delta = \inf_{c\in A}\mathcal{C}_{\text{CODING}}(\lambda,c) - \mathcal{C}_{\text{CODING}}(\lambda,c_o).$$

Then

$$\begin{aligned}\mathcal{C}_{\text{CODING}}(\lambda,c_o) &= \inf_{c\in A}\mathcal{C}_{\text{CODING}}(\lambda,c) - \Delta \\ &\leq \mathcal{C}_{\text{CODING}}(\lambda,c_1) - \Delta \\ &< \mathcal{C}_{\text{CODING}}(\lambda,c_2).\end{aligned}$$

But this is impossible since $\mathbf{C}_{\text{CODING}}(\lambda,c_o) \geq \mathbf{C}_{\text{CODING}}(\lambda,c_2)$ for $c_2 \in B$. So (A3.2) is untrue. Therefore

$$\inf_{c \in A} \mathbf{C}_{\text{CODING}}(\lambda,c) = \mathbf{C}_{\text{CODING}}(\lambda,c_o).$$

Lemma: Consider the Poisson channel with parameters $\lambda_o(t)$, $c(t)$. For all $\Delta_b > 0$ there exists a nonnegative simple function $\lambda_b(t)$ dominated by $\lambda_o(t)$ such that

$$|\mathbf{C}_{\text{CODING}}(\lambda_b, c + \lambda_o - \lambda_b) - \mathbf{C}_{\text{CODING}}(\lambda_o, c)| < \Delta_b .$$

Proof: Let A be the set of simple functions dominating $\lambda_o(t)$ and let B be the set of nonnegative simple functions dominated by $\lambda_o(t)$). Let $\lambda_a \in A$, $\lambda_b \in B$, and define $\delta_a(t) = \lambda_a(t) - \lambda_o(t)$, $\delta_b(t) = \lambda_o(t) - \lambda_b(t)$. $\delta_a(t)$, $\delta_b(t)$ are both nonnegative. Divert a portion, $\delta_a(t) \wedge c(t)$, from $c(t)$ to produce noise. Then

$$\mathbf{C}_{\text{CODING}}(\lambda_o, c) \geq \mathbf{C}_{\text{CODING}}(\lambda_o + (\delta_a \wedge c), c - (\delta_a \wedge c)) = \mathbf{C}_{\text{CODING}}(\lambda_a, c - (\delta_a \wedge c)).$$

Likewise

$$\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) \geq \mathbf{C}_{\text{CODING}}(\lambda_o, c).$$

Thus we have

$$|\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) - \mathbf{C}_{\text{CODING}}(\lambda_o, c)| \leq |\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) - \mathbf{C}_{\text{CODING}}(\lambda_a, c - (\delta_a \wedge c))| .$$

Fix $\varepsilon > 0$. To complete the proof we show that there exists $\lambda_a \in A$, $\lambda_b \in B$, such that

$$|\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) - \mathbf{C}_{\text{CODING}}(\lambda_a, c - (\delta_a \wedge c))| < \varepsilon . \tag{A3.3}$$

Write

$$\begin{aligned} &|\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) - \mathbf{C}_{\text{CODING}}(\lambda_a, c - (\delta_a \wedge c))| \\ &\quad \leq |\mathbf{C}_{\text{CODING}}(\lambda_b, c + \delta_b) - \mathbf{C}_{\text{CODING}}(\lambda_b, c)| \\ &\quad + |\mathbf{C}_{\text{CODING}}(\lambda_b, c_a) - \mathbf{C}_{\text{CODING}}(\lambda_b, c)| \\ &\quad + |\mathbf{C}_{\text{CODING}}(\lambda_b, c_a) - \mathbf{C}_{\text{CODING}}(\lambda_b, c_b)| \\ &\quad + |\mathbf{C}_{\text{CODING}}(\lambda_a, c) - \mathbf{C}_{\text{CODING}}(\lambda_a, c_b)| \\ &\quad + |\mathbf{C}_{\text{CODING}}(\lambda_a, c) - \mathbf{C}_{\text{CODING}}(\lambda_a, c - (\delta_a \wedge c))| \end{aligned} \tag{A3.4}$$

where c_a and c_b are simple functions such that

$$c(t) - (\delta_a(t) \wedge c(t)) \leq c_b(t) \leq c(t) \leq c_a(t) \leq c(t) + \delta_b(t). \tag{A3.5}$$

Let $c_a(t) \to c(t)$, $c_b(t) \to c(t)$. Then by the previous lemma, the second and fourth differences on the RHS of (A3.4) decrease to zero. Let $\lambda_a(t) \to \lambda(t)$, $\lambda_b(t) \to \lambda(t)$ consistent with (A3.5). Then the third difference on the RHS of (A3.4) decreases to zero since all the channel parameters present in the

expression of the difference are simple. Also, by the previous lemma, the first and fifth differences decrease to zero. Thus (A3.3) is satisfied and the proof is complete.

Proposition: Let $\lambda_1(t)$ and $c_1(t)$ be the parameters of a Poisson channel and let $\lambda_2(t)$ and $c_2(t)$ be the parameters of a second Poisson channel. Suppose $\lambda_1(t)$ is dominated by $\lambda_2(t)$ and $c_1(t)$ dominates $c_2(t)$. Then

$$\mathcal{C}_{\text{CODING}}(\lambda_1, c_1) \geq \mathcal{C}_{\text{CODING}}(\lambda_2, c_2).$$

Proof: Choose $\lambda_{1b}(t)$ and $\lambda_{2b}(t)$ to be nonnegative simple functions dominated, respectively, by $\lambda_{1b}(t)$ and $\lambda_2(t)$. Write

$$\begin{aligned}
&\mathcal{C}_{\text{CODING}}(\lambda_1, c_1) - \mathcal{C}_{\text{CODING}}(\lambda_2, c_2) \\
&\qquad = \mathcal{C}_{\text{CODING}}(\lambda_1, c_1) - \mathcal{C}_{\text{CODING}}(\lambda_1, c_2) \\
&\qquad - \Delta_1 + \Delta_2 + \Delta_3 \\
&\qquad + \mathcal{C}_{\text{CODING}}(\lambda_{1b}, c_2 + \lambda_2 - \lambda_{2b}) - \mathcal{C}_{\text{CODING}}(\lambda_{2b}, c_2 + \lambda_2 - \lambda_{2b})
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &= \mathcal{C}_{\text{CODING}}(\lambda_{1b}, c_2 + \lambda_1 - \lambda_{1b}) - \mathcal{C}_{\text{CODING}}(\lambda_1, c_2), \\
\Delta_2 &= \mathcal{C}_{\text{CODING}}(\lambda_{2b}, c_2 + \lambda_2 - \lambda_{2b}) - \mathcal{C}_{\text{CODING}}(\lambda_2, c_2), \\
\Delta_3 &= \mathcal{C}_{\text{CODING}}(\lambda_{1b}, c_2 + \lambda_1 - \lambda_{1b}) - \mathcal{C}_{\text{CODING}}(\lambda_{1a}, c_2 + \lambda_2 - \lambda_{2b}).
\end{aligned}$$

By the previous lemma, $\lambda_{1b}(t)$, $\lambda_{2b}(t)$ can be chosen so that $\lambda_{1b}(t) \leq \lambda_{2b}(t)$ with Δ_1, Δ_2, and Δ_3 arbitrarily close to zero. For $\lambda_{1b}(t) \leq \lambda_{2b}(t)$,

$$\mathcal{C}_{\text{CODING}}(\lambda_{1b}, c_2 + \lambda_2 - \lambda_{2b}) - \mathcal{C}_{\text{CODING}}(\lambda_{2b}, c_2 + \lambda_2 - \lambda_{2b}) \geq 0.$$

Also

$$\mathcal{C}_{\text{CODING}}(\lambda_1, c_1) \geq \mathcal{C}_{\text{CODING}}(\lambda_1, c_2).$$

Therefore

$$\mathcal{C}_{\text{CODING}}(\lambda_1, c_1) \geq \mathcal{C}_{\text{CODING}}(\lambda_2, c_2).$$

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